ON GLOBAL EXISTENCE OF LOCALIZED SOLUTIONS OF SOME NONLINEAR LATTICES

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ABSTRACT

We prove global existence and uniqueness of solutions of some important nonlinear lattices which include the Fermi-Pasta-Ulam (FPU) lattice. Our result shows (on a particular example) that the FPU lattice with high nonlinearity and its continuum limit display drastically different behaviour with respect to blow up phenomenon.

Key words: Nonlinear lattices, energy method, high interaction.

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1 Introduction

In recent years nonlinear differential-difference equations, also called lattices, attracted a great deal of attention. Various aspects of spatially localized solutions of classical and quantum lattices with high nonlinearity (i.e. nonlinearity characterized by powers greater than four) were discussed in a number of recent papers (see e.g. [1, 2, 14] and references therein). Related questions are of traditional interest in nonlinear physics because on the one hand lattices serve as mathematical models for many physical and biological systems like, for example, coupled oscillators, macromolecules, spin chains, etc. On the other hand, the discreteness introduces in evolution many new qualitative features compared with the respective continuum limits. Among them we can mention Bloch oscillations of bright [6, 8] and dark [9] solitons in the so-called Ablowitz-Ladik model with a linear potential, dynamical localization of a soliton in a periodic potential [8] sensitivity of the lattice dynamics to the type of the nonlinearity [10]. One of the most important questions in the lattice dynamics is the *existence* of various types of solutions. An important step in this direction has recently been made in [1, 15] where the proof of existence and stability of breathers, i.e. solutions periodic in time and localized in space, has been reported. Those papers dealt with a special limit, called anticontinuous, when the coupling between two neighbour sites is small enough (i.e. when on-site potentials dominate inter-site interactions). Meanwhile the existence and stability of localized excitations in more relevant physical situations when the coupling in nonlinear lattices is of a general type remains an open question (unlike the behaviour of various nonlinear continuum models, see e. g. [3, 4, 5]). If the model considered appears with some kind of dissipative mechanism once the global existence it is settle then questions especially important for physical applications are the stability and possible uniform rates of decay of the solutions at time approaches infinity. Results in this direction will appear in a forthcoming work [16].

The purpose of the present paper is to prove global existence (and uniqueness) of a solution of an important nonlinear lattice, the so-called Fermi-Pasta-Ulam (FPU) lattice. In order to obtain our results we built a convenient Banach space and get local solutions via a contraction. The calculations are lengthly because we work with estimates of the explicit Green function associated to the problem. We prefer it in this way because it becames clear in the proofs that the techniques we use work as well for many other important models with suitable modifications (see section 4). The lattices are assumed to be described by Hamiltonians of the following type

$$\mathcal{H} = \mathcal{H}_\ell + \mathcal{H}_{n\ell} \tag{1}$$

where

$$\mathcal{H}_\ell = \frac{M}{2}\sum_{n=-\infty}^{\infty} \dot{u}_n^2 + \frac{K_2}{2}\sum_{n=-\infty}^{\infty} (u_{n+1}-u_n)^2\,, \tag{2}$$

with $u_n \equiv u_n(t)$ being a displacement of the nth particle, $M > 0$ being the mass of a particle, K_2 being a force constant and $\dot{u}_n$ denotes the derivative with respect to time. $\mathcal{H}_\ell$ is the linear part describing nearest neighbour interactions. The nonlinear part $\mathcal{H}_{n\ell}$ we will consider is given by

$$\mathcal{H}_{n\ell} = \frac{K_{p+1}}{p+1} \sum_{n=-\infty}^{\infty} (u_{n+1} - u_n)^{p+1} . \tag{3}$$

where p is an integer, $p \geq 2$ and K_{p+1} is a positive constant.

The paper is organized as follows: In Sections 2 and 3, we give detailed proofs of local and global existence of spatially localized solutions of the FPU lattice. In Section 4 we present some other important lattices for which the techniques apply as well with suitable modifications, for example the "sine-Gordon" inter-site nonlinearity where the nonlinear part is given by

$$\mathcal{H}_{n\ell} = \mathcal{H}_{SG-IS} = K \sum_{n=-\infty}^{\infty} [1 - \cos(u_{n+1} - u_n)] \tag{4}$$

and the so-called on-site nonlinearity

$$\mathcal{H}_{n\ell} = \mathcal{H}_{OS} = \frac{K_{p+1}}{p+1} \sum_{n=-\infty}^{\infty} u_n^{p+1} \tag{5}$$

2 Local solutions of the FPU lattice

The evolution equation for the FPU lattice reads

$$M\ddot{u}_n = K_2(u_{n+1} + u_{n-1} - 2u_n) - K_{p+1} \left[(u_n - u_{n-1})^p + (u_n - u_{n+1})^p \right] \tag{6}$$

All considerations below are restricted to the case when p is odd and M, K_2, and K_{p+1} are positive constants.

After renormalization $\sqrt{K_2/M}t \ \mapsto t$ the equation of the motion of the lattice (1)-(3) is written in the form

$$\ddot{\alpha}_n(t) - \Delta\alpha_n(t) - \chi_p \Delta\alpha_n^p(t) = 0 \tag{7}$$

where $n \in \mathbb{Z}$, $\alpha_n(t) = u_{n+1} - u_n$ is a relative displacement, $\chi_p = K_{p+1}/K_2$, and $\Delta f_n \equiv f_{n+1} + f_{n-1} - 2f_n$. We notice that introducing new variable $\alpha_n(t)$ makes not only physical sense allowing to include kink-like solutions in the consideration. The energy in the linearized problem for $\alpha_n(t)$ can be used as a norm (see below) while the energy in the linearized problem for $u_n(t)$ can be treated only as a seminorm.

We consider a Cauchy problem for equation (7) subject to the initial conditions

$$\alpha_n(t=0) = \tilde{a}_n = a_{n+1} - a_n; \qquad \dot{\alpha}_n(t=0) = \tilde{b}_n = b_{n+1} - b_n \tag{8}$$

which correspond to "physical" initial data for the displacement field of the form

$$u_n(t=0) = a_n, \qquad \dot{u}_n(t=0) = b_n \tag{9}$$

Note that (7), (8) can be (formally) rewritten in the integral form

$$\alpha_n(t) = \beta_n(t) + \chi_p \sum_{m=-\infty}^{\infty} \int_0^t G_1\left(n-m, t-s\right)\ \alpha_m^p(s) ds \tag{10}$$

$$\alpha_n(t) = \beta_n(t) + \chi_p \sum_{m=-\infty}^{\infty} \int_0^t G\left(n-m, t-s\right)\ \Delta\alpha_m^p(s) ds \tag{11}$$

where $\beta_n(t)$ is a solution of the Cauchy problem

$$\begin{cases} \ddot{\beta}_n(t) = \Delta\beta_n(t) \\ \beta_n(t=0) = \tilde{a}_n; \qquad \dot{\beta}_n(t=0) = \tilde{b}_n \\ \lim_{n\to\infty} \beta_n(t) = 0 \end{cases} \tag{12}$$

$$G(n,t) = \frac{2}{\pi} \int_{-\pi}^{\pi} \cos\left(2\sigma\ n\right) \frac{\sin\left(2t \sin\ \sigma\right)}{\sin\ \sigma}\, d\sigma, \tag{13}$$

is the Green function associated with (12) and

$$G_1(n,t) = \frac{2}{\pi} \int_{-\pi}^{\pi} \cos\left(2\sigma n\right) \sin\left(2t \sin\ \sigma\right)\ \sin\ \sigma\, d\sigma, \tag{14}$$

with $n = 0, \pm 1, \pm 2, \cdots$. We recall some elementary properties of the functions $G_1(n,t)$ and $G(n,t)$ which we will use in what follows

$$|G_1(0,t)| \leq 8\,|t|, \tag{15}$$

$$|G_1(n,t)| \leq C\ \frac{|t| + |t|^3}{n^2}, \qquad \forall n \neq 0,\ \forall t \in I\!R. \tag{16}$$

$$|\dot{G}(0,t)| \leq 8, \tag{17}$$

$$|\dot{G}(n,t)| \leq C\ \frac{|t|^2}{n^2}, \qquad \forall n \neq 0,\ \forall t \in I\!R. \tag{18}$$

Hereafter various positive constants will be denoted as C but, we remark that they may vary from line to line.

Let $T > 0$. We consider the linear space $H = H(T)$ (it will be refered to as the energy space) which consists of all functions $\alpha(t)$ of the form:

$$\alpha(t) = (\cdots, \alpha_{j-1}(t), \alpha_j(t), \alpha_{j+1}(t), \cdots)$$

such that

$$\text{(a)} \qquad \text{each} \qquad \alpha_j(t) \in C^2([0,T);\ I\!R)$$

$$\text{(b)} \qquad \sup_{0\le t<T} \left[\sum_{n=-\infty}^{+\infty} \left(\sum_{m=-\infty}^{n-1} \dot{\alpha}_m(t) \right)^2 + \sum_{n=-\infty}^{+\infty} \alpha_n^2(t) \right] < +\infty.$$

We define the norm in H as

$$||\alpha(\cdot)||_H^2 = \sup_{0\le t<T} \left[\sum_{n=-\infty}^{+\infty} \left(\sum_{m=-\infty}^{n-1} \dot{\alpha}_m(t) \right)^2 + \sum_{n=-\infty}^{+\infty} \alpha_n^2(t) \right] \tag{19}$$

and $(H, ||\cdot||_H)$ becomes a Banach space. Thus H is the space of spatially localized functions $\alpha_n(t)$ having finite energy. The respective localization law is implicitly defined by (19).

Lemma 1 *Let $\beta(t) = (\cdots, \beta_{j-1}(t), \beta_j(t), \beta_{j+1}(t), \cdots)$ where $\beta_n(t)$ is the solution of the Cauchy problem (12). Assume that $\{\tilde{a}_n\}$ and $\{\tilde{b}_n\}$ satisfy the condition*

$$\sum_{n=-\infty}^{+\infty} \left(\sum_{m=-\infty}^{n-1} |\tilde{b}_m|^2 \right)^2 + \sum_{n=-\infty}^{+\infty} |\tilde{a}_n|^2 < \infty. \tag{20}$$

Then, for any $T > 0$ the function $\beta(t)$ belongs to the energy space $H = H(T)$

Proof: Lemma 1 follows from the explicit representation of the solution $\beta_n(t)$

$$\beta_n(t) = \sum_{m=-\infty}^{+\infty} \left[\dot{G}(n-m,t)\, \tilde{a}_m + G(n-m,t)\, \tilde{b}_m \right]$$

and the properties of the Green function. $\square$

Let $R > 0$. We define the subset

$$F_R = \left\{ \alpha(t) \in H, \quad ||\alpha(\cdot) - \beta(\cdot)||_H \le R; \quad \alpha_n(0) = \beta_n(0), \quad \dot{\alpha}_n(0) = \dot{\beta}_n(0), \ \ \forall n \right\} \subseteq H.$$

Clearly, F_R is a closed subset of H.

Theorem 1 *(Local Existence). Assume that $\{\tilde{a}_n\}$ and $\{\tilde{b}_n\}$ are as in Lemma 1. Then, there exists $T > 0$ and a unique function*

$$\alpha(t) = (\cdots, \alpha_{j-1}(t), \alpha_j(t), \alpha_{j+1}(t), \cdots) \tag{21}$$

defined in $[0, T)$, such that $\alpha_j(t)$ is a solution of (7), (8), and $\alpha(t)$ belongs to the Banach space H.

Proof: Let us consider the map

$$P\alpha(t) = \left(\cdots, \widetilde{P}\alpha_{j-1}(t), \widetilde{P}\alpha_j(t), \widetilde{P}\alpha_{j+1}(t), \cdots\right) \tag{22}$$

where

$$\begin{aligned} \widetilde{P}\alpha_n &= \beta_n(t) + \chi_p \sum_{m=-\infty}^{+\infty} \int_0^t G_1(n-m, t-s)\, \alpha_m^p(s) ds = \\ &= \beta_n(t) + \chi_p \sum_{m=-\infty}^{+\infty} \int_0^t G(n-m, t-s)\, \Delta\alpha_m^p(s) ds, \qquad (23) \\ &\qquad p \in \mathbb{Z}^+, \;\; p \geq 2, \;\; 0 \leq t < T. \end{aligned}$$

It is useful to note that

$$\widetilde{P}\alpha_n(t=0) = \tilde{a}_n, \qquad \frac{d}{dt}\, \widetilde{P}\alpha_n(t=0) = \tilde{b}_n. \tag{24}$$

The proof consists of two steps: (a) First, we show that $P: \; F_R \mapsto F_R$ if T is chosen sufficiently small. (b) For the second step we show that P is contraction on F_R, which means that there exists γ, $0 < \gamma < 1$, such that

$$||Pf - Pg||_H \leq \gamma ||f-g||_H; \qquad \forall f, g \in F_R$$

provided T is chosen sufficiently small.

Due to Lemma 1, in order to show that $P\alpha \in H$ whenever $\alpha \in H$ it is sufficient to show that $P\alpha - \beta \in H$. To this end we consider the two terms in the energy norm separately and start with the "potential energy". It follows from (23) that

$$\sum_{n=-\infty}^{+\infty} \left[\widetilde{P}\alpha_n(t) - \beta_n(t)\right]^2 = \chi_p^2 (A_1 + A_2 + A_3) \tag{25}$$

where

$$A_1 = \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \sum_{\substack{\ell=-\infty \\ \ell\neq n}}^{+\infty} \int_0^t G_1(n-m, t-s)\, \alpha_m^p(s) ds \int_0^t G_1(n-\ell, t-s_1)\, \alpha_\ell^p(s_1)\, ds_1 \tag{26}$$

$$A_2 = 2 \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \int_0^t G_1(n-m, t-s)\, \alpha_m^p(s) ds \int_0^t G_1(0, t-s_1)\, \alpha_n^p(s_1)\, ds_1 \tag{27}$$

$$A_3 = \sum_{n=-\infty}^{+\infty} \left(\int_0^t G_1(0, t-s)\, \alpha_n^p(s) ds \right)^2 . \tag{28}$$

Using the fact that

$$\sum_{n=-\infty}^{+\infty} |\alpha_n(s)|^p |\alpha_n(s_1)|^p \leq ||\alpha||_H^{2p} \tag{29}$$

it is straightforward to obtain that

$$\sup_{0 \leq t < T} |A_3| \leq C ||\alpha||_H^{2p}\, T^4 . \tag{30}$$

In order to estimate $|A_1|$ we first use (16), what yields

$$|A_1| \leq C \sum_{n=-\infty}^{+\infty} \left(\sum_{k \neq 0} \int_0^t \frac{|t-s| + |t-s|^3}{k^2}\, |\alpha_{n-k}^p(s)|\, ds \right)^2 . \tag{31}$$

Next, we take into account that $p \geq 2$ and employ (29) to get

$$\sup_{0 \leq t < T} |A_1| \leq C ||\alpha||_H^{2p}\, (T^2 + T^4)^2 . \tag{32}$$

Similar calculations to the above ones lead to

$$\sup_{0 \leq t < T} |A_2| \leq C ||\alpha||_H^{2p}\, (T^2 + T^4)\, T^2 . \tag{33}$$

Combining (30)-(33) we get the estimate for the "potential energy"

$$\sup_{0 \leq t < T} \sum_{n=-\infty}^{+\infty} \left[\widetilde{P} \alpha_n(t) - \beta_n(t) \right]^2 \leq C ||\alpha||_H^{2p}\, (T^2 + T^4)^2 . \tag{34}$$

Next we calculate

$$\frac{d}{dt}\, \widetilde{P} \alpha_n(t) - \dot{\beta}_n(t) = \sum_{m=-\infty}^{+\infty} \int_0^t \dot{G}(m, t-s)\, \Delta \alpha_{m+n}^p(s) ds . \tag{35}$$

It is easy to verify that the relation

$$\sum_{m=-\infty}^{n-1} \sum_{\ell=-\infty}^{+\infty} \dot{G}(m-\ell, t-s)\, \Delta \alpha_\ell^p(s) =$$

$$= \sum_{\ell=-\infty}^{+\infty} \left[\dot{G}(m-\ell, t-s) - \dot{G}(n-1-\ell, t-s) \right] \left[\alpha_{\ell+1}^p(s) - \alpha_\ell^p(s) \right] \tag{36}$$

holds. Consequently, we only need to estimate

$$\sum_{n=-\infty}^{+\infty}\left(\sum_{\ell=-\infty}^{+\infty}\int_0^t \dot{G}(n-\ell-1,t-s)\left[\alpha_{\ell+1}^p(s)-\alpha_\ell^p(s)\right]ds\right)^2 \leq B_1+B_2 \tag{37}$$

where

$$B_1 = 2\sum_{n=-\infty}^{+\infty}\left(\sum_{\ell\neq n-1}^{+\infty}\int_0^t \dot{G}(n-\ell-1,t-s)\left[\alpha_{\ell+1}^p(s)-\alpha_\ell^p(s)\right]\,ds\right)^2$$

and

$$B_2 = 2\sum_{n=-\infty}^{\infty}\left(\int_0^t \dot{G}(0,t-s)\left[\alpha_n^p(s)-\alpha_{n-1}^p(s)\right]ds\right)^2.$$

Using (17), (18) we estimate B_1 and B_2 as follows

$$|B_2| \leq CT^2||\alpha(\cdot)||_H^{2p} \tag{38}$$

and

$$|B_1| \leq CT^6||\alpha(\cdot)||_H^{2P}. \tag{39}$$

Finally, we have that

$$\begin{aligned} ||P\alpha-\beta||_H^2 &\leq C||\alpha||_H^{2p}\left(T^4+T^6+T^8\right) \\ &\leq C(R+||\beta(0)||)_H^{2p}\left(T^4+T^6+T^8\right)\leq R \end{aligned} \tag{40}$$

if T is chosen sufficiently small. Thus, $P:\ F_R \mapsto F_R$ which shows (a).

Now we show that P is a contraction map from F_R into F_R if T is chosen small enough. Consider $\alpha(t)$, $\delta(t)\in H$. We can write

$$\sup_{0\leq t<T}\sum_{n=-\infty}^{+\infty}\left[\widetilde{P}\alpha_n(t)-\widetilde{P}\delta_n(t)\right]^2 = \chi_p^2(C_1+C_2+C_3) \tag{41}$$

where

$$C_1 = \sup_{0\leq t<T}\sum_{n=-\infty}^{+\infty}\left(\int_0^t G_1(0,t-s)\left[\alpha_n(s)-\delta_n(s)\right]\left[\alpha_n^{p-1}(s)+\cdots+\delta_n^{p-1}(s)\right]ds\right)^2 \tag{42}$$

$$\begin{aligned} C_2 = \sup_{0\leq t<T}\sum_{n=-\infty}^{+\infty}&\left(\int_0^t G_1(0,t-s)\left[\alpha_n(s)-\delta_n(s)\right]\left[\alpha_n^{p-1}(s)+\cdots+\delta_n^{p-1}(s)\right]ds\right)\times \\ &\times\left(\sum_{\substack{\ell=-\infty\\ \ell\neq n}}^{+\infty}\int_0^t G_1(\ell-n,t-s_1)\left[\alpha_\ell(s_1)-\delta_\ell(s_1)\right]\left[\alpha_\ell^{p-1}(s_1)+\cdots+\delta_\ell^{p-1}(s_1)\right]ds_1\right) \end{aligned} \tag{43}$$

and

$$C_3 = \sup_{0\le t<T} \sum_{n=-\infty}^{+\infty} \left[\int_0^t \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} G_1(n-m, t-s) \left[\alpha_m(s) - \delta_m(s)\right] \times \right.$$
$$\left. \times \left[\alpha_m^{p-1}(s) + \cdots + \delta_m^{p-1}(s)\right] ds \right]^2. \tag{44}$$

Straightforward estimates using (15), (16) and similar to those provided above yield

$$\begin{aligned} |C_1| &\le C||\alpha(\cdot) - \delta(\cdot)||_H^2 T^4 \left(||\alpha(\cdot)||_H + ||\delta(\cdot)||_H\right)^{2(p-1)} \\ &\le C||\alpha(\cdot) - \delta(\cdot)||_H^2 T^4 R^{2(p-1)} \end{aligned} \tag{45}$$

$$|C_2| \le C||\alpha(\cdot) - \delta(\cdot)||_H^2 (T^4 + T^5) \left(2||\beta(0)||_H + 2R\right)^{2(p-1)} \tag{46}$$

$$|C_3| \le C||\alpha(\cdot) - \delta(\cdot)||_H^2 (T^5 + T^6) \left(2||\beta(0)||_H + 2R\right)^{2(p-1)} \tag{47}$$

Therefore

$$\begin{aligned} &\sup_{0\le t<T} \sum_{n=-\infty}^{+\infty} \left[\widetilde{P}\alpha_n(t) - \widetilde{P}\delta_n(t)\right]^2 \le \\ &\le C(2||\beta(0)||_H + 2R)^{2(p-1)} (T^4 + T^5 + T^6) ||\alpha(\cdot) - \delta(\cdot)||_H^2 \le \\ &\le C||\alpha(\cdot) - \delta(\cdot)||_H^2. \end{aligned} \tag{48}$$

Now we can estimate the "kinetic" term as follows: Let us write

$$\sup_{0\le t<T} \sum_{n=-\infty}^{+\infty} \left(\sum_{m=-\infty}^{+\infty} \frac{d}{dt} \left[\widetilde{P}\alpha_m(t) - \widetilde{P}\delta_m(t)\right] \right)^2 = \chi_p^2 (D_1 + D_2 + D_3) \tag{49}$$

with

$$D_1 = \sup_{0\le t<T} \sum_{n=-\infty}^{+\infty} \left(\int_0^t \dot{G}(0, t-s) \left[\alpha_n^p(s) - \delta_n^p(s) - \alpha_{n-1}^p(s) + \delta_{n-1}^p(s)\right] ds \right)^2 \tag{50}$$

$$\begin{aligned} D_2 \quad &= \sup\nolimits_{0\le t<T} \textstyle\sum_{n=-\infty}^{+\infty} \left(\int_0^t G(0, t-s) \left[\alpha_n^p(s) - \delta_n^p(s) - \alpha_{n-1}^p(s) + \delta_{n-1}^p(s)\right] ds \right) \times \\ &\times \left(\textstyle\sum_{\substack{\ell=-\infty \\ \ell\neq n}}^{+\infty} \int_0^t \dot{G}(\ell, t-s_1) \left[\alpha_{n+\ell}^p(s_1) - \delta_{n+\ell}^p(s_1) + \delta_{n+\ell-1}^p(s_1)\right] ds_1 \right) \end{aligned} \tag{51}$$

and

$$D_3 = \sup_{0\le t<T} \sum_{n=-\infty}^{+\infty} \left[\sum_{\substack{\ell=-\infty \\ \ell\neq n}}^{+\infty} \int_0^t \dot{G}(\ell, t-s) \left[\alpha_{n+\ell}^p(s) - \delta_{n+\ell}^p(s) - \alpha_{n+\ell-1}^p(s)\right] ds \right]^2. \qquad (52)$$

In the same way as (45)–(47) have been obtained but now using (17) and (2) we get

$$D_1 \qquad \le C||\alpha(\cdot) - \delta(\cdot)||_H^2 \,(2||\beta(0)||_H + 2R)^{2(p-1)} \qquad (53)$$

$$D_2 \quad \le C||\alpha(\cdot) - \delta(\cdot)||_H^2 \,(T^2 + T^3)\,(2||\beta(0)||_H + 2R)^{2(p-1)} \qquad (54)$$

$$D_3 \quad \le C||\alpha(\cdot) - \delta(\cdot)||_H^2 \,(T^3 + T^4)\,(2||\beta(0)||_H + 2R)^{2(p-1)} \qquad (55)$$

Combining the last three inequalities with (49) and taking $T > 0$ small enough so that

$$C(2||\beta(0)||_H + 2R)^{2(p-1)}(T^2 + T^3 + T^4 + T^5 + T^6) < 1$$

then P is a contraction in F_R. Consequently, we have a unique fixed point $\widetilde{\alpha} \in F_R$, i.e., $P\widetilde{\alpha} = \widetilde{\alpha}$. This means that

$$\widetilde{P}\widetilde{\alpha}_j(t) = \widetilde{\alpha}_j(t), \qquad \forall t \in [0, T) \qquad (56)$$

i.e.

$$\widetilde{\alpha}_j(t) = \beta_j(t) + \sum_{m=-\infty}^{+\infty} \int_0^t G_1(j-m, t-s)\, \widetilde{\alpha}_m^p(s) ds. \qquad (57)$$

where $T > 0$ is chosen as above. $\square$

Remark 1. We conclude this section mentioning that the norm we have chosen above, having rather transparent physical meaning, is not the only allowing us to prove the local existence. We could have use for example a Banach space X of functions

$$\alpha(t) = (..., \alpha_{j-1}(t), \alpha_j(t), \alpha_{j+1}(t), ...)$$

such that $\alpha_j \in C^1([0,T); \boldsymbol{R})$ and

$$||\alpha(\cdot)||_X^2 = \sup_{-\infty<n<\infty} (1+n^2) \left(||\alpha_n(\cdot)||_\infty^2 + \left\| \sum_{m=-\infty}^{n-1} \dot{\alpha}_m(\cdot) \right\|_\infty^2 \right) \qquad (58)$$

where

$$||f_n(\cdot)||_\infty = \sup_{0\le t<T} |f_n(t)|$$

and for each $n \in \boldsymbol{Z}$, $\alpha_n(t)$ satisfies

$$\sup_{0\le t<T} |\alpha_n(t)| < \infty, \qquad \sup_{0\le t<T} \left| \sum_{m=-\infty}^{n-1} \dot{\alpha}_m(t) \right| < \infty.$$

In $\alpha(t) \in X$ then

$$\alpha_n^2(t) \le \frac{||\alpha(\cdot)||_X^2}{1+n^2}, \quad \left(\sum_{m=-\infty}^{n-1} \dot{\alpha}_m(t)\right)^2 \le \frac{||\alpha(\cdot)||_X^2}{1+n^2}, \qquad \forall n, \ \ \forall t \in [0,T)$$

which implies that $||\alpha(\cdot)||_H^2 \le C||\alpha(\cdot)||_X^2$ where $\|\cdot\|_H$ was defined in (21).

3 Global solution

In order to prove global existence we will use a conserved quantity of the problem at hand. In terms of the depending variables $\alpha_n(t)$ one of the conservation quantities is the Hamiltonian $\mathcal{H}_\ell + \mathcal{H}_{nl}$ itself. In fact, let

$$E(t) = \sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{2} \left(\sum_{m=-\infty}^{n-1} \dot{\alpha}_m(t)\right)^2 + \frac{1}{2}\,\alpha_n^2(t) + \frac{\chi_p}{p+1}\,\alpha_n^{p+1}(t) \right\} \tag{59}$$

where $\alpha_m(t)$ is the solution found in Theorem 1. If we assume that

$$\sum_{n=-\infty}^{+\infty} |\tilde{a}_n|^{p+1} < +\infty$$

then we can show that $E(t)$ is finite for any t in the interval of existence of $\alpha_n(t)$. Straightforward calculation using the fact that $\alpha_n(t)$ solves (7)-(8) shows that $\frac{dE}{dt} = 0$. Consequently, $E(t)$=constant.

In order to discuss consequences of the conservation of the energy we first notice that Theorem 1 gives the local existence of localized excitations in FPU lattice in terms of the relative displacements $\alpha_n(t)$. Then, the question on the existence of the solution in terms of original displacements of the particles arises. It is resolved by the following lemma:

Lemma 2 *Let $\{\tilde{\alpha}_n\}$ and $\{\tilde{b}_n\}$ be as in Lemma 1 and $\alpha = (\cdots, \alpha_n(t), \cdots)$ be the finite energy solution of the Cauchy problem (7), (8) with $p \ge 2$. Then, in the interval of existence of $\alpha_n(t)$ the function*

$$\tilde{u}_n(t) = \sum_{m=-\infty}^{n} \alpha_m(t) \tag{60}$$

is well defined and it is bounded.

Proof: Substitution of α_m by the expression in (10), using the fact that $\sum_{n=-\infty}^{+\infty} |\alpha_n(t)|^{p+1} < +\infty$ with $p \ge 2$ and $\alpha \in H$ for $t \in [0,T)$ proves the lemma. $\square$

The solution of the FPU lattice is obtained from the trivial relation

$$u_n(t) = \tilde{u}_n(t) + \text{const}$$

We know that Zorn's lemma [7] implies that the local solution $\alpha(t)$ we found above can be extended to the maximal interval of existence $0 \le t < T_{\max}$. We want to show that $T_{\max} = +\infty$.

Theorem 2 *(Global Existence). Let $p \ge 3$ be an odd integer and $\chi_p > 0$. Suppose that the initial conditions for problem (7), (8) satisfy the assumptions of Lemma 1 and*

$$\sum_{n=-\infty}^{+\infty} \tilde{a}_n^{p+1} < +\infty. \tag{61}$$

Then the finite energy solution $\alpha(t)$ exists globally and it is unique.

Proof: We consider an interval $0 \le T < T_{\max}$ where T could be very near to $T_{\max}$. Let

$$\alpha(t) = (\cdots, \alpha_{j-1}(t), \alpha_j(t), \alpha_{j+1}(t), \cdots)$$

be the solution of (7), (8) in the norm of H given in (21).

Global existence in the energy norm follows as a consequence of the conservation of $E(t)$ given in (59), the fact that p is an odd integer greater than or equal to 3 and $\chi_p > 0$.

It remains to show that the global solution in unique. To this end suppose that $\alpha(t)$ and $\delta(t)$ are two solutions of (7), (8) with the same initial conditions at $t = 0$. Let $T > 0$ be fixed but arbitrary. Introduce

$$v_n(t) = \sum_{m=-\infty}^{n-1} \alpha_m(t), \qquad w_n(t) = \sum_{m=-\infty}^{n-1} \delta_m(t).$$

Due to Lemma 2, $v_n(t)$ and $w_n(t)$ are well defined and bounded in any interval. Clearly, $v_n(0) = w_n(0)$ and $\dot{v}_n(0) = \dot{w}_n(0)$. We claim that

$$z_n(t) = v_n(t) - w_n(t) \equiv 0, \qquad \forall 0 \le t \le T.$$

Denote by

$$f_n(t) = [v_n(t) - v_{n-1}(t)]^{p-1} + \cdots + [w_n(t) - w_{n-1}(t)]^{p-1}.$$

Then equation for $z(t)$ takes the form

$$\begin{cases} \ddot{z}_n - \Delta z_n = \chi_p \left[(z_{n+1} - z_n) f_{n+1} - (z_n - z_{n-1}) f_n \right] \\ z_n(t=0) = \dot{z}(t=0) = 0. \end{cases} \tag{62}$$

Clearly the following relations

$$\frac{1}{2}\,\frac{d}{dt}\sum_{n=-\infty}^{\infty}\left\{\dot{z}_n^2(t)+\left[z_{n+1}(t)-z_n(t)\right]^2\right\}=$$

$$=\chi_p\sum_{n=-\infty}^{\infty}\dot{z}_n(t)\left[\left(z_{n+1}-z_n\right)f_{n+1}-\left(z_n-z_{n-1}\right)f_n\right]. \tag{63}$$

holdds and using Cauchy-Schwarz's inequality we deduce that

$$\left|\sum_{n=-\infty}^{\infty}\dot{z}_n(t)\left[\left(z_{n+1}-z_n\right)f_{n+1}-\left(z_n-z_{n-1}\right)f_n\right]\right|\leq$$

$$\leq 2\sup_{-\infty<n<\infty}|f_n(t)|\left(\sum_{n=-\infty}^{\infty}\right)^{\frac{1}{2}}\left(\sum_{n=-\infty}^{\infty}\left(z_{n+1}-z_n\right)^2\right)^{\frac{1}{2}}.$$

Next, we take into account that

$$\sup_{-\infty<n<\infty}||\alpha_n(\cdot)||_{\infty}^{p-1}\leq||\alpha(\cdot)||_H^{p-1},$$

which implies that $\sup_{-\infty<n<\infty}|w_n|<\infty$. Returning to the relation (63) we obtain that

$$\frac{d}{dt}\,\varphi(t)\leq C\sup_{-\infty<n<\infty}|f_n(t)|\,\varphi(t)\leq C_1\varphi(t) \tag{64}$$

where

$$\varphi(t)=\sum_{n=-\infty}^{\infty}\left\{\dot{z}_n^2(t)+\left[z_{n+1}(t)-z_n(t)\right]^2\right\}\geq 0.$$

From (64) and Gronwall's lemma we deduce that $\varphi(t)$ is identically zero. $\square$

Remark 2. The FPU lattice considered in this section if of great practical importance and that is why received much attention in physical literature where it has been treated with help of numerous approximations. One of the approaches, the so-called long-wavelength limit (or continuum limit) is often used in order to reduce the system of differential-difference equations to an evolution partial differential equation, namely the generalized Korteweg-de Vries (GKdV) equation

$$\frac{\partial v}{\partial t}+\frac{\partial^3 v}{\partial x^3}+\chi_p v^{p-1}\,\frac{\partial v}{\partial x}=0. \tag{65}$$

with $-\infty<x<+\infty$, $t>0$. It is well known that (65) for p large enough displays a dispersive blow up. This was shown by Bona and Saut in [5] in special situations but in general, for $p>5$ there is only numerical evidence that blow up really happens. We remark that this difficulty does not occur in the discrete case treated above which allows us to study strong interactions.

4 Some other important models

In this section we shall consider some additional models for which the global existence (and uniqueness) of the solution can be found in the same way we proceed in the previous sections. First, we consider the sine-Gordon lattice with nonlinear inter-site interactions

$$\mathcal{H}_{nl} = K \sum_{n=-\infty}^{\infty} [1 - \cos(u_{n+1} - u_n)] \tag{66}$$

and linear part is given by (2). The dynamical equations in terms of the relative displacements $\alpha_n(t)$ read

$$\ddot{\alpha}_n - \Delta\alpha_n - \chi\Delta \sin\ \alpha_n = 0. \tag{67}$$

The only difference compared with (7) consists on type of nonlinearity. Hence we can continue using the space H we considered in Section 2 with the norm (19) and use Lemma 1. Since $|\sin\ f| \leq |f|$ one reduces the problem at hand to the considerations given in Theorem 1 to obtain local solutions. In order to obtain global existence we consider the function

$$E(t) = \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{2} \left(\sum_{m=-\infty}^{n-1} \dot{\alpha}_m(t) \right)^2 + \frac{1}{2}\ \alpha_n^2(t) + \chi[1 - \cos\ \alpha_n(t)] \right\}$$

Using (67) it is eassy to prove that $\frac{dE}{dt} = 0$. Therefore

$$E(t) = \text{constant} = E(0) \tag{68}$$

Thus, whenever the constant χ is positive then, (68) will tell us that the solution $\alpha(t) = (..., \alpha_{j-1}(t), \alpha_j(t), \alpha_{j+1}(t), ...)$ remains bounded in the norm of the space H, which proves that the solution exist globally. Uniqueness can be shown as we proceeded in Theorem 2.

Next we consider the so-called a lattice with on-site nonlinearity. In this case $\mathcal{H}_l$ is as in (2) and

$$\mathcal{H}_{n\ell} = \mathcal{H}_{OS} = \frac{K_{p+1}}{p+1} \sum_{n=-\infty}^{\infty} u_n^{p+1}$$

The dynamical equations read

$$\ddot{u}_n - \Delta u_n + \chi_p\ u_n^p = 0 \tag{69}$$

In this case it is convenient to explore again the integral form (12). To prove local existence we define for $t > 0$ the linear space $\tilde{H} = \tilde{H}(T)$ which consists of all functions

$$u(t) = (\cdots, u_{j-1}(t), u_j(t), u_{j+1}(t), \cdots)$$

such that

$$\text{(a)} \qquad u_j(t) \in C^2\left([0,T_0);\ I\!R\right)$$

and

$$\text{(b)} \qquad \sup_{0\le t<T} \sum_{n=-\infty}^{\infty} \left[\dot{u}_n^2(t) + u_n^2(t)\right] < +\infty.$$

The energy norm in $\tilde{H}$ is defined as

$$||u(t)||_{\tilde{H}}^2 = \sup_{0\le t<T} \left\{ \sum_{n=-\infty}^{\infty} \left[\dot{u}_n^2(t) + u_n^2(t)\right] \right\}.$$

and $(\tilde{H}, \|\cdot\|_{\tilde{H}})$ is a Banach space. we can easily show that if

$$\beta(t) = (\cdots, \beta_{j-1}(t), \beta_j(t), \beta_{j+1}(t), \cdots)$$

where $\beta_j(t)$ is the solution of (14) with

$$\sum_{n=-\infty}^{\infty} \left(|\tilde{a}_n|^2 + |\tilde{b}_n|^2\right) < +\infty \tag{70}$$

then, the function $\beta(t)$ belongs to the space $\tilde{H}$ for any $T > 0$.

Under the above assumptions we can prove local existence of (69) if T is chosen sufficiently small. Furthermore, if we assume additionally that

$$\sum_{n=-\infty}^{+\infty} |\tilde{a}_n|^{p+1} < +\infty$$

we can consider the function

$$E(t) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} [\dot{u}_n^2(t) + u_n^2(t)] + \frac{\chi_p}{p+1} \sum_{n=-\infty}^{+\infty} u_n^{p+1}(t) \tag{71}$$

and show that $E(t)$ is finite for any t in the interval of existence. Using (69) we can prove that $\frac{dE}{dt} = 0$, therefore $E(t)$ =constant= $E(0)$ for any t in the interval of existence. hence, if we assume that $\chi_p > 0$ and p =odd≥ 3 we obtain the desired a priori estimate for the norm $\|u(\cdot)\|_{\tilde{H}}$ which proves global existence. Uniqueness can be shown as in Theorem 2. Similar discussions can be done for the so-called sine-Gordon on-site nonlinear lattice

$$\ddot{u}_n - \Delta u_n + \chi \sin u_n = 0.$$

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